Concours Général de Mathématiques «Minko Balkanski»

Solutions

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Solution 1.

The inequality can be rewritten as

$$a+b+c > ab+bc+ac.$$

Consider the polynomial $P(x) = (x-a)(x-b)(x-c)$. We know that $P(0) = -1 < 0$ and $P(1) = ab+bc+ac-(a+b+c) < 0$. In particular, none of a, b, c is equal to 1 and an even number among the them are in $(0, 1)$. Yet, $abc = 1$ gives that this number is not 0 (otherwise $a = b = c = 1$). This concludes the proof. For the converse, note that if P has two roots in the interval $(0, 1)$, the third is larger than 1, and therefore $P(1) < 0$.

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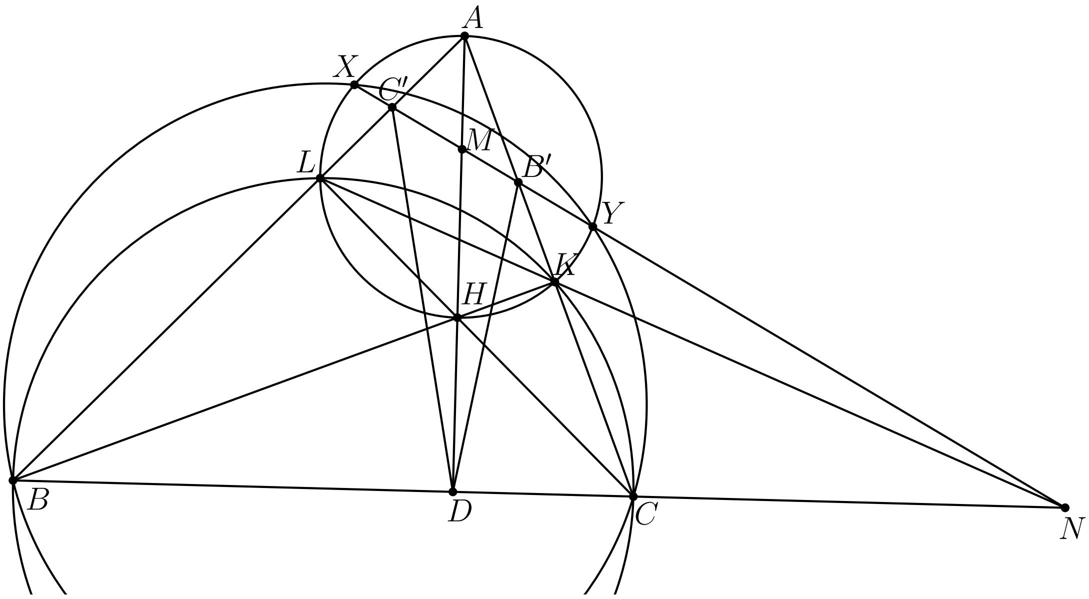
Solution 2.



Let K and L be the feet of the altitudes from B and C in triangle ABC, respectively, and let AD intersect XY at M. Then, K and L lie on both k_1 and k_2. Therefore, by the radical axis theorem for k_1, k_2 and the circle around $BCKL$ we have that XY, KL and BC intersect at one point.

At the same time, B, D, C and N form a harmonic quadruple, and, by projecting from A, one may deduce that the same holds for C', M, B' and N. However, $\angle MDN = 90°$, and therefore DA bisects angle $B'DC'$, as desired.

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Solution 3.

$\boxed{\text{Only when } n = 1 \text{ or } k \in \{1, 2\}.}$ Let N be the total number of eleventh grade students. Label the conspiring students $1, \ldots, n$ and let $L_i \subset \{1, \ldots, N\}$ be the list of student i for $i \in \{1, \ldots, N\}$.

If $n = 1$, there is nothing to do.

If $k = 1$, it suffices to set $L_i = \{i + 1\}$ for $i < n$ and $L_n = \{1\}$.

If $k = 2$, we set $L_1 = \{2, 3\}$, $L_2 = \{1, 3\}$, $L_i = \{1, 2\}$ for $i \in [3, n]$. (Here, we used that $N > k = 2$ as otherwise it is impossible for the students to form their lists.)

Assume that $k \geqslant 3$. Further, let $n < k + 1$. Then $L_1 \backslash \{1, \ldots, n\}$ contains some $i \in \{n + 1, \ldots, N\}$. But it may happen that i lists 1, in which case the principal can form the class $\{1, i\}$ and $\{2, \ldots, N\} \backslash \{i\}$. Thus, the students don't have a winning strategy.

Let us assume that, on the contrary, $n \geqslant k + 1$, and consider the oriented graph associated to the lists. We may assume that $L_i = \{1, \ldots, k\}$ for all $i > n$. We claim that if the associated (directed) graph G has two vertex-disjoint (directed) cycles, we are done. Indeed, these cycles necessarily contain some of the students $1, \ldots, n$ (since others have in-degree 0), so we may put each cycle in a different class and divide the remaining vertices into two groups: those connected to the first cycle via a directed path and the remaining ones.

It therefore remains to show that every directed graph of minimum out-degree at least 3 has two vertex-disjoint cycles. Assume the contrary and let G be a counter-example with minimum number of edges. Notice that any graph of minimum out-degree at least 1 has a cycle. Therefore, if G has a cycle of length 2, the remainder of G has a cycle, so we obtain a contradiction. Therefore, all cycles have length at least 3.

We show that for each vertex $x \in G$ there is a cycle C in the in-neighbourhood of x. To begin with, note that if x has zero in-degree, deleting x contradicts the minimality of G. Now, for any edge yx, we consider the graph G' obtained by deleting the out-edges of y and contracting the edge yx. Notice that G' does not have two vertex-disjoint cycles, so it must have out-degree less than 3. In particular, there exists z such that both zy and zx are edges in G. This means that the subgraph of G induced by the in-neighbourhood of x has minimum in-degree at least 1, so it contains a cycle, as claimed.

Moreover, reversing the edges of G yields another minimal counter-example, so there is also a cycle C' contained in the out-neighbourhood of x in G. However, G has no cycles of length 2, and therefore C and C' are disjoint, thus concluding the proof.

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