Concours Général de Mathématiques «Minko Balkanski»

Solutions

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Solution 1.

Suppose without loss of generality that the leading coefficient of $P(x) = 3f(x) - 7g(x) - 2h(x)$ is positive (if not, multiply f, g and h by -1). Set $Q(x) = f(x) + g(x) - 4h(x)$ and $R(x) = f(x) + g(x) + h(x)$. Resolving the system of linear equations leads to

$$\begin{aligned} 10f(x) &= P(x) + Q(x) + 6R(x), \\ 10g(x) &= -P(x) + Q(x) + 2R(x), \\ 10h(x) &= - 2Q(x) + 2R(x). \end{aligned}$$

Thus, one may conclude from the first two equalities that $Q(x)$ and $R(x)$ could not have leading coefficients with the same sign, while the third equality shows that $h(x)$ is either always non-positive or always non-negative.

Hence, since $h(0) = 0$, 0 must also be the unique root of $Q(x)$ and $R(x)$, so $f(0)+g(0) = 2Q(0) + 8R(0) = 0$.

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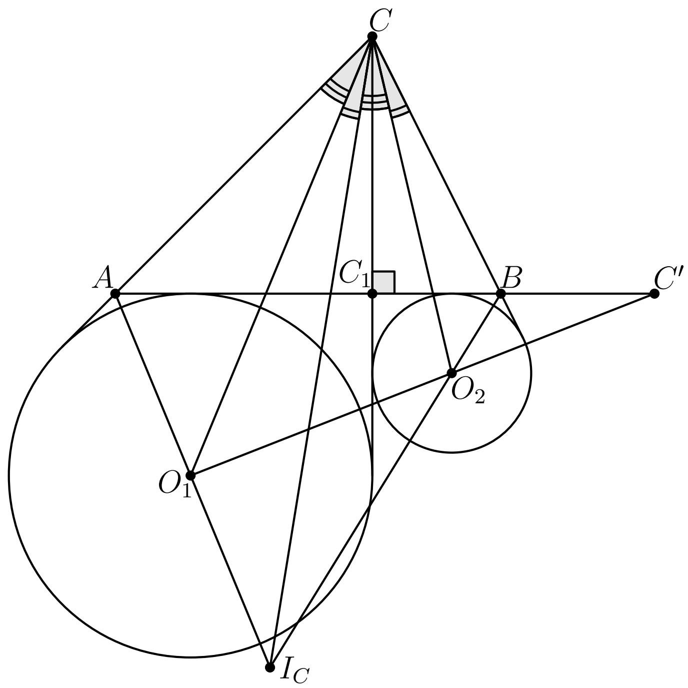

Solution 2.

Denote by O_1 and O_2 the centers of c_1 and c_2 respectively, and by I_C the center of the escribed circle of triangle ABC, opposite to the vertex C. By the Menelaus theorem for the triangle ABI_C and the line through O_1, O_2 and C',

$$\frac{|AC'|}{|BC'|} = \frac{|AO_1| \cdot |I_CO_2|}{|O_1I_C| \cdot |O_2B|}.$$

Hence, by the law of sines, iterated 4 times for the triangles AO_1C, O_1I_CC, I_CO_2C and O_2BC respectively,

$$\begin{aligned}
\frac{|AO_1| \cdot |I_CO_2|}{|O_1I_C| \cdot |O_2B|} &= \frac{|AO_1|}{|O_1C|}\frac{|O_1C|}{|O_1I_C|}\frac{|I_CO_2|}{|O_2C|}\frac{|O_2C|}{|O_2B|} \\
&= \frac{\sin\angle ACO_1}{\sin\angle O_1AC}\frac{\sin\angle CI_CO_1}{\sin\angle O_1CI_C}\frac{\sin\angle I_CCO_2}{\sin\angle O_2I_CC}\frac{\sin\angle CBO_2}{\sin\angle O_2CB} \\
&= \frac{\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)}\frac{\sin\frac{\beta}{2}}{\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}\frac{\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}\frac{\sin\left(\frac{\pi}{2}+\frac{\beta}{2}\right)}{\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)} \\
&= \frac{(\sin\beta)\cdot\left(\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right)^2}{(\sin\alpha)\cdot\left(\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right)^2} \\
&= \frac{(\sin\beta)\cdot\left(1-\cos\left(\frac{\pi}{2}-\alpha\right)\right)}{(\sin\alpha)\cdot\left(1-\cos\left(\frac{\pi}{2}-\beta\right)\right)} \\
&= \frac{(\sin\beta)\cdot(1-\sin\alpha)}{(\sin\alpha)\cdot(1-\sin\beta)} = \frac{1-(\sin\alpha)^{-1}}{1-(\sin\beta)^{-1}},
\end{aligned}$$

where we used the formula $(\sin x)^2 = \frac{1-\cos(2x)}{2}$.

Similar expressions for $\frac{|BA'|}{|A'C|}$ et $\frac{|CB'|}{|B'A|}$ lead to

$$\frac{|AC'|}{|BC'|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|} = \frac{1-(\sin\alpha)^{-1}}{1-(\sin\beta)^{-1}}\frac{1-(\sin\beta)^{-1}}{1-(\sin\gamma)^{-1}}\frac{1-(\sin\gamma)^{-1}}{1-(\sin\alpha)^{-1}} = 1,$$

which suffices to conclude by the inverse theorem of Menelaus.

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Solution 3.

Firstly, note that for any two graphs $G_1 \subseteq G_2$ we have $s(G_1) \leqslant s(G_2)$ (we call this property *monotonicity*). Consequently, it suffices to prove the desired inequality for bipartite G with parts V_1 and V_2, $|V_1| = 2^{k_1}s_1 + 1$ and $|V_2| = 2^{k_2}s_2 + 1$ for two integers $k_1, k_2 \geqslant 0$. Indeed, if V_1 is not of this form, i.e. $|V_1| \in [2^{k_1}s_1 + 2, 2^{k_1+1}s_1]$ for an integer $k_1 \geqslant 0$, we remove $|V_1| - (2^{k_1}s_1 + 1)$ arbitrarily chosen elements of V_1 and similarly for V_2; we also delete $V \backslash (V_1 \cup V_2)$.

We argue by induction on $N = \min\{k_1, k_2\}$ that $s(G) \geqslant N+1$, which implies the desired inequality by monotonicity. For $N = 0$, we have $s(G) \geqslant 1$, since G contains at least two edges. Indeed, at least one edge v_1v_2 connects V_1 and V_2, and at least one connects $V_1 \backslash \{v_1\}$ and $V_2 \backslash \{v_2\}$. Assume that the induction hypothesis is verified for $N = N_0 - 1$. Let $\mathcal{F}$ be a good family of orderings of V of size $r = s(G)$. Fix a $\sigma \in \mathcal{F}$ and let

$$t = \min_{1 \leqslant j \leqslant |V|} \left\{ |\sigma^{-1}([1,j]) \cap V_1| = 2^{k_1-1}s_1 + 1 \text{ or } |\sigma^{-1}([1,j]) \cap V_2| = 2^{k_2-1}s_2 + 1 \right\}.$$

Roughly speaking, t is the first index such that $\sigma^{-1}([1,t])$ contains more than half of V_1 and less than half of V_2 or vice versa. Without loss of generality, let $|\sigma^{-1}([1,t]) \cap V_1| = 2^{k_1-1}s_1 + 1$. Set $\sigma^{-1}([1,t]) \cap V_1 = U_1$ and $V_2 \backslash \sigma^{-1}([1,t]) = U_2$. Up to removing $|U_2| - (2^{k_2-1}s_2 + 1) \geqslant 2^{k_2}s_2 + 1 - 2^{k_2-1}s_2 - (2^{k_2-1}s_2 + 1) \geqslant 0$ elements from U_2, we get U_1 and U_2 satisfying the induction hypothesis for $N_0 - 1$. Thus,

$$|\mathcal{F} \backslash \{\sigma\}| = r - 1 \geqslant s(G[U_1 \cup U_2]) \geqslant N_0 - 1,$$

since σ doesn't separate any couple of edges of $G([U_1 \cup U_2])$. This concludes the proof.

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