Concours Général de Mathétmatiques "Minko Balkanski"

Solutions

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Solution 1.

Fix $\varepsilon \in (0, 1/8)$. Then there exists N_ε such that for every $n \geqslant N_\varepsilon$ we have

$$a_{n+1} > a_n + a_n^2 - \varepsilon.$$

Observe that if $N \geqslant N_\varepsilon$ and $|a_N| \geqslant \sqrt{2\varepsilon}$, then $a_{N+1} > a_N + a_N^2 - \varepsilon \geqslant a_N + \varepsilon$. Let us assume that such an N exists for some $\varepsilon \in (0, 1/8)$. By induction, this yields $a_{N+k} \geqslant a_N + k\varepsilon$, as long as $|a_{N+m}| > \sqrt{2\varepsilon}$ for all $m \in \{0, 1, \dots, k-1\}$. If $a_N > \sqrt{2\varepsilon}$ we obtain that $\lim_{n\to+\infty} a_n = +\infty$.

Hence, it remains to treat the case that $a_n \leqslant \sqrt{2\varepsilon}$ for all $n \geqslant N_\varepsilon$. Then necessarily there exists $k > N_\varepsilon$ such that $|a_k| \leqslant \sqrt{2\varepsilon}$. But then by induction for all $m > k$ it holds that $a_m \geqslant -\sqrt{2\varepsilon} + \varepsilon$. Indeed, if $a_m \geqslant -\sqrt{2\varepsilon}$, then

$$a_{m+1} \geqslant \min_{x \in [-\sqrt{2\varepsilon}, \sqrt{2\varepsilon}]} x + x^2 - \varepsilon = -\sqrt{2\varepsilon} + \varepsilon,$$

as $\varepsilon < 1/8$.

To sum up, we obtained that for each $\varepsilon \in (0, 1/8)$ one of three things occurs:

- $\lim_{n\to+\infty} a_n = +\infty$;
- for all m large enough $-\sqrt{2\varepsilon} + \varepsilon \leqslant a_m \leqslant \sqrt{2\varepsilon}$.

Clearly, this implies that, if the first possibility fails, then $\lim_{n\to+\infty} a_n = 0$.

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Solution 2.

Let us begin by making a few common observations. Firstly, each bike can only run the distance from A to B once (going back is counted with negative sign). Therefore, if there are n people and k bikes, a total distance of $n|AB|$ is travelled and at most $k|AB|$ of it is ridden. Hence, the total time spent doing this by the company is at least

$$\left(\frac{k}{20} + \frac{n-k}{5}\right)|AB|,$$

so, given that we are interested in the time needed for the last person to reach B, a lower bound is

$$\left(\frac{k}{20} + \frac{n-k}{5}\right)\frac{|AB|}{n}. \tag{1}$$

Furthermore, in order for this to be achieved, all friends need to arrive at B simultaneously, ride for $|AB|k/(20n)$ hours, walk the rest of the distance and never stop or go back.

a. $\boxed{\text{15 minutes.}}$ The lower bound follows from the general reasoning above. Let us produce an example. It suffices for Marius to ride the bike for the first half of the distance and walk the rest. Ivo walks the first half, picks the bike up and rides the rest. Both of them take $1/20 + 1/5 = 1/4$ hours, that is. 15 minutes.

b. $\boxed{\left(\frac{k}{20} + \frac{n-k}{5}\right)\frac{|AB|}{n} \text{ hours.}}$ The lower bound was already established, so we turn to the example. Let us number the people from 1 to n and split the distance from A to B in n equal parts, whose endpoints we call checkpoints, so that B is checkpoint n. The first person rides a bike to checkpoint k and walks to B. Person 2 rides to checkpoint $k-1$, walks to checkpoint $n-1$, picks up a bike there and finishes the distance on bike. The third person rides to checkpoint $k-2$, walks to $n-2$, from where to n he uses a bike found at $n-2$. We continue in the same way, rotating the configuration cyclically. The last person first walks to checkpoint 1, then rides to $k+1$ and walks to n.

Let us check that this is indeed valid, namely that there is a bike for each person to pick up (it is clear that, if this is the case, the time each person takes is given by (1)). Indeed, each person needs to pick up a bike once except the first person and this happens at a different checkpoint for each person, while each person leaves a bike at a different checkpoint 1 through n (with the convention that person $n-k+1$ leaves it at n). It remains to observe that the person leaving the bike indeed arrives at the corresponding checkpoint before the one picking it up. But this is clear, since the person picking a bike up either rides it all the way to the end or has been walking from the beginning. In both cases that person is the last one in the group.

There are many other ways to achieve the optimal time.

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Solution 3.

$\boxed{\text{There are no solutions to the equation.}}$ Suppose without loss of generality that $a \geqslant b$ and $p \geqslant q$, and denote $X = \frac{a!+b!}{(p-1)!+(q-1)!}$ and $Y = \frac{3(n^2+1)}{2}$. Observe that Y, depending on the parity of n, Y is either an odd multiple of 3 or an odd multiple of 3 divided by 2 (which we also call odd).

Assume that $b \geqslant q+1$**.** Then, modulo $4(q-1)!$ the numerator of X is 0, while its denominator is $(q-1)!$ or $2(q-1)!$, so X cannot be odd—contradiction.

Assume that $b = q$**.** Then q divides $3(n^2+1)$, so $b = q = 3$, since n^2+1 has no prime divisors of the form $4k+3$. This contradicts $12|q-7$.

Assume that $b = q-1$**.** For the numerator of X to be divisible by 3 we need $a = q$ or $a = q+1$. But then necessarily $p = q$, since otherwise $p \geqslant q+12$ and $X < 1$. Thus, we arrive at $\frac{a!}{(p-1)!} = 3n^2+2$, which is a contradiction modulo 4 (considering $a = p-1$ separately).

Assume that $b \leqslant q-2$**.** Clearly, $a \geqslant q-1$, since otherwise $X \leqslant 1$ and $Y \geqslant 3/2$. Then $(q-1)!$ divides the denominator of X, while the numerator is congruent to $b!$ modulo $(q-1)!$, so $\frac{(q-1)!}{b!}|2$ and, thus, $q = 3 > b$. Since 3 has to divide $a!+b!$, an easy check shows that there are no solutions.

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