Concours Général de Mathétmatiques "Minko Balkanski"

Solutions

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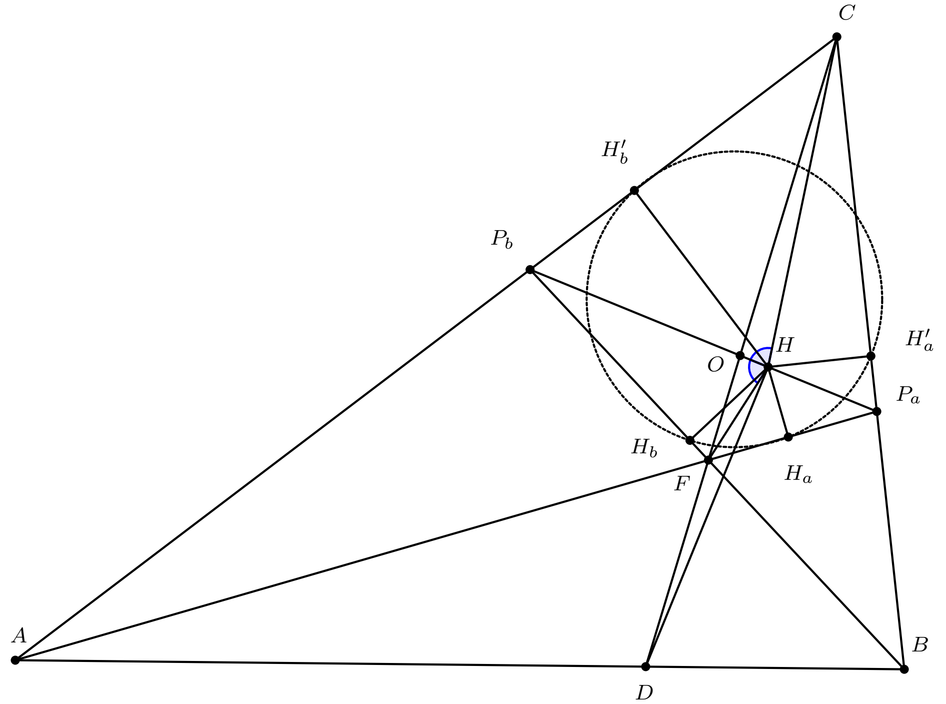
Solution 1.

Let $O = FC \cap P_aP_b$. We have that:

- $\angle HH'_aH_a = \angle HP_aH_a = \angle HP_aF$.
- $\angle HH'_aH'_b = \angle HCH'_b = \angle HCP_b$.
- $\angle HH_bH_a = \angle HFH_a = \angle HFP_a$.
- $\angle HH_bH'_b = \angle HP_bH'_b = \angle HP_bC$.

Therefore, the sum of angles $H_aH'_aH'_b$ and $H_aH_bH'_b$ is indeed equal to $180° - \angle P_bHC + \angle P_bHF$. We prove that $\angle P_bHC = \angle P_bHF$.

Remark that the lines (BA, BP_b, BO, BC) form a harmonic quadruple. Projecting on the line CD gives that (D, F, O, C) is a harmonic quadruple. Now, define F' to be the point on the line CD, for which $\angle OHF' = OHC$ and $F' \neq C$. Then, OH is an internal bisector of $\angle FHC$ and therefore $F'O/OC = FH/HC$. Moreover, $\angle OHD = 90°$ so HD is an external bisector of $\angle FHC$. Therefore $F'D/DC = F'D/DC = F'O/OC$. We conclude that (D, F', O, C) is a harmonic quadruple, so $F' \equiv F$. This completes the proof.



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Solution 2.

a. We prove the statement by induction. It therefore suffices to notice that

$$\frac{1}{2^n+1}+\cdots+\frac{1}{2^{n+1}} \geqslant \underbrace{\frac{1}{2^{n+1}}+\cdots+\frac{1}{2^{n+1}}}_{2^n}=\frac{1}{2}.$$

b. $\boxed{\lim 1/u_n = \lim 1/v_n = 0.}$ Notice that by induction both u_n and v_n are positive and therefore increasing. This implies the existence of the limits.

Let us assume for a contradiction that $u_n \to c < \infty$. Then $u_n \geqslant c/2$ for all $n \geqslant n_0$ for some n_0 sufficiently large. Therefore, $v_{n+1} \leqslant v_n + \frac{4040}{c}$ for $n \geqslant n_0$, so, by induction

$$v_n \leqslant v_{n_0} + (n-n_0)\frac{4040}{c}.$$

Plugging this in the recurrence relation for u_n, we get that for $n \geqslant n_0$

$$u_n \geqslant u_{n_0} + \sum_{i=0}^{n-n_0} \frac{8}{v_{n_0}+i\frac{4040}{c}} \geqslant \frac{c}{505}\sum_{i=N}^{N+n-n_0}\frac{1}{i},$$

where $N = \lceil c v_{n_0}/4040 \rceil$.

Denote $H_n = \sum_{i=1}^{n}\frac{1}{i}$. From **a.** we have that $H_n \to \infty$. Hence

$$u_n \geqslant \frac{c}{505}(H_{N+n-n_0} - H_{N-1}) \overset{n\to\infty}{\longrightarrow} \infty.$$

This contradicts the hypothesis $c < \infty$ and concludes the proof that $\lim u_n = \infty$. The divergence of v_n can be proved similarly or using that, by induction $v_n \geqslant u_n$.

Remark. The curious reader may study $\lim u_n/\sqrt{n}$ when 2020 is replaced by 8 in the statement of the problem.

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Solution 3.

$\boxed{n \geqslant k.}$ Observe that the order in which the devil places tokens does not matter. It is easily checked that if Thomas places k consecutive tokens, the devil will keep adding tokens to their left to infinity.

Let A be the set of (integers occupied by the) tokens of Thomas and B the devil's tokens. Assume that B is infinite. Without loss of generality we may assume that B contains infinitely many negative integers.

Observation 1. Assume that for some $z \in B$ at some point in during step 1. the devil places a token on $z-k$ or $z-k+1$, so that a token on z is already present. Then a token either on $z-k$ or on $z-(k-1)$ is placed after the one on z, so the token on z is also placed during step 1. Therefore, the tokens on $z+k$ and $z+k-1$ are placed before z.

Necessarily, there exists $b \in B$ with $b < \min A - k^2$, on which the token is placed in step 1. By Observation 1 the tokens on $b+k < \min A$ and $b+k-1 < \min A$ are placed during step 1. before the one on b. Iterating this argument we obtain that the tokens on $X_0 = \{b+k^2-k+1, \ldots, b+k^2\}$ are all placed in step 1. Notice that $\max X_0 < \min A$ by our choice of b. Without loss of generality we assume that $b = k - k^2$.

For $i \geqslant 0$ let $X_i = \{ki+1, \ldots, k(i+1)\}$ and let Y_i be the set of integers $x \in X_i$ on which tokens were placed in step 1. In other words, X_i partition the positive integers into blocks of k consecutive ones and Y_i are the integers in block i where tokens are placed during step 1. In particular, we have that $Y_0 = X_0$.

Observation 2. If for some i and y we have that $y \in Y_{i-1}$ and $y+k \notin Y_i$, then $y+k \in A$.

Proof. By definition of Y_{i-1} we have that a token was placed on $y+k$ before y. By Observation 1 applied to $y+k$ we get that if $y+k \notin A$, then the token on $y+k$ was placed in step 1., contradicting $y+k \notin Y_i$. □

We consider two possibilities. Assume first that B contains a finite number of positive integers. Then, setting $i_0 = \max(A \cup B)$ it is clear that $Y_{i_0} = \varnothing$, as no tokens were ever placed in X_{i_0}. If, on the contrary, B contains an integer $b' > \max A + k^2 + k$ on which the token is placed in step 2., then as above all tokens on $\max A + 1, \ldots, \max A + 2k$ are placed in step 2. Then, setting $i_0 = \lceil (\max A)/k \rceil$, we have $Y_{i_0} = \varnothing$, as all tokens in X_{i_0} are placed in step 2.

Recalling that $|Y_0| = k$ and $|Y_{i_0}| = 0$, we get from Observation 2 that A contains at least one integer in each residue class modulo k, so $|A| \geqslant k$.

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