Concours Général de Mathétmatiques "Minko Balkanski"

Solutions

18 May 2019

Solution 1.

a. Let $n = 101$. We begin by giving a lower bound for the minimal time of evacuation and then construct an algorithm attaining it. Consider floor number i. The number of people waiting for the elevator on a higher floor is initially

$$\sum_{s=i+1}^{101} s = \frac{101.102}{2} - \frac{i.(i+1)}{2} = \frac{(101-i)(102+i)}{2}. \tag{1}$$

Thus, the number of times the elevator has to go up to take people from floors $i+1$, $i+2\ldots$ up to 101 is

$$\left\lceil \frac{(101-i)(102+i)}{10} \right\rceil. \tag{2}$$

To obtain a lower bound for the time the elevator spends moving between the floors, we sum (2) from $i = 0$ to $i = 100$ and multiply by four seconds (two for going up and two for going down). In order to bound the time spent waiting on floors, notice that the elevator has to wait at least $8\left\lceil \frac{i}{5} \right\rceil$ seconds on the floor number i.

We will give an example meeting both bounds, so having a total evacuation time

$$4\sum_{i=0}^{100}\left\lceil \frac{(101-i)(102+i)}{10} \right\rceil + 8\sum_{i=1}^{101}\left\lceil \frac{i}{5} \right\rceil = 287572$$

seconds or 79 hours, 52 minutes and 52 seconds. To achieve this, one can perform the following procedure.

- Bring down 5 people from a floor as long as possible (in any order). After this on floor i the number of people left is the remainder of i modulo 5.
- Bring down the remaining people 5 by 5 by pairing successive non-empty floors together – $5k+6$ with $5k+4$, $5k+3$ with $5k+2$ for $0 \leqslant k \leqslant 19$.
- Bring down the only person on the first floor.

The elevator stops the minimal number of times, $\left\lceil \frac{i}{5} \right\rceil$, on floor i in order to collect all the people. In order to treat the travelling time, we prove that that for each floor $i \leqslant 100$ the distance between floors i and $i+1$ is run the minimal number of times. Notice that for each such link between floors the number of ascends is equal to the number of descents, as the elevator starts and finishes at floor 0. Moreover, each link is travelled downwards exactly the number of times in (2) (check this separately for floor numbers with different remainders modulo 5).

b. For $n = 100$, we repeat a similar algorithm, whose correctness is proved by the same method (the verification for the travelling time is slightly different). The modified algorithm is as follows.

- Bring down 5 people from a floor as long as possible (in any order). After this on floor i the number of people left is the remainder of i modulo 5.

- Bring down the remaining people 5 by 5 by pairing non-empty floors together – $5k+4$ with $5k+1$, $5k+3$ with $5k+2$ for $0 \leqslant k \leqslant 19$.

The time is 2 hours, 20 minutes and 12 seconds less.

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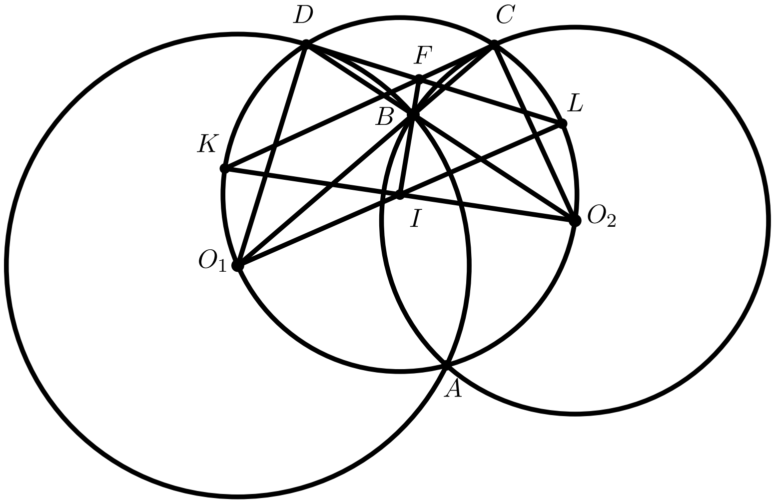
Solution 2.



First notice that $\angle O_1CO_2 = \angle O_1DO_2 = 180° - \angle O_1BO_2 = 180° - \angle O_1AO_2$. From this observation we get that O_1, O_2, A, C and D lie on a common circle. Let the line DF intersect this circle at L, different from D, and also let the line CF intersect this circle at K, different from C. We have that O_1L and O_2K are diameters in k and thus intersect at I. We conclude from Pascal's theorem for the circle k and the triplets of points D, K, O_1 and C, L, O_2.

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Solution 3.

a. First notice that $\boxed{\text{constant functions}}$ are solutions. Assume that the function is non-constant. Plugging $n = 1$ in the relation, one obtains

$$\forall m, k \in \mathbb{N}, f(m+1)|f(m) + f(1)^{f(k)}.$$

We consider two cases.
Case 1. Assume that $f(1) > 1$. Then the statement above gives

$$\forall m, k, s \in \mathbb{N}, f(m+1) \mid f(1)^{f(s)} - f(1)^{f(k)}.$$

As f is not constant, it follows that for s, k such that $f(s) \neq f(k)$, $f(m+1) \mid f(1)^{f(s)} - f(1)^{f(k)}$ and in particular f is bounded. Then for any N large enough, $f(m+N^k) = f(N) = \max f$. But the divisibility condition for $n = N$ and any m, k gives in this case that $\max(f) \mid f(m)$. In consequence, f must be a constant function, which was already treated.
Case 2. Assume that $f(1) = 1$. Let $a = \min\{n \in \mathbb{N} | f(n) > 1\}$. The divisibility condition for $m = k = 1$ and $n = a - 1$ gives that $f(a) = 2$. Now, $f(a+1) \mid f(a) + f(1) = 3$

and if $a \geqslant 3$, we also have $f(a+1) \mid f(2) + f(a-1) = 2$. This imposes $f(a+1) = 1$, contradiction with the assumption that f is non-decreasing. So $a = 2$ and $f(2) = 2$. An easy induction is sufficient to conclude that in this case we obtain the $\boxed{\text{identity function}}$, which is indeed a solution of the problem.

b. Setting $k = 1$ in the divisibility condition, we obtain that

$$f(m+n) \mid f(m) + f(n). \tag{3}$$

By induction one obtains $f(n) \leqslant n$ for all n. We consider two cases.
Case 1. Assume that $f(2) = 1$. Let $a = \min\{n \in \mathbb{N} | f(n) > 1\} > 2$, as above. As in **a.** one obtains that $f(a) = 2$. Then, as in **a.**, by induction $f(n) = 1$ for all n not divisible by a and $f(n) \mid 2$ for all n divisible by a. Indeed, by (3) we have $f(n) \mid f(a) + f(n-a)$ and $f(n) \mid f(\alpha) + f(n-\alpha)$ for $\alpha < a$ with $a \nmid n - \alpha$ and the two sums can be computed by induction hypothesis.

Note that if $f(la) = 1$ for some l, then $f(n) = 1$ for all $n \geqslant la$, since $f(ma) \mid f(a) + f((m-1)a) = 3$ by induction and we already know that $f(ma) \mid 2$, so f is equivalent to the $\boxed{\text{constant function } 1}$ and we are done. But the only other possibility is to have $f(n) = 2$ if $a \mid n$ and 1 otherwise. We claim that this is possible if and only if $a = \prod p_i$ for distinct primes p_i. To see that those are solutions, note that $a \mid m + n^k$ implies that for all i we have m and n are either both divisible or both not divisible by p_i. Thus, $f(m + n^k) = 2$ implies that $f(m) = f(n)$ and the desired divisibility holds. Hence,

$$\boxed{f(n) = 2 \text{ if } a \mid n \text{ and } 1 \text{ otherwise with } a = \prod p_i \text{ for } p_i \text{ distinct primes}}$$

is a solution. Assume that $p^2 \mid a$ for some prime p. Then $2 = f(a + (a/p)^2) = f(a) + f(a/p)^{f(2)} = 3$ – a contradiction.
Case 2. Assume that $f(2) = 2$. Let $n_k = \min\{n > n_{k-1} | f(n_k) = 1\}$ with $n_1 = 1$.

Note that if for some k we have $n_k - n_{k-1} = 1$, then $f(n) = 1$ for all $n \geqslant n_{k-1}$, which is equivalent to the constant 1. Assume that $n_k - n_{k-1} \geqslant 2$ for all k. If $n_k - n_{k-1} = 2$ for some k, then $f(n_k - 2) = 1 = f(n_k)$, $f(n_k - 1) = 2 = f(n_k + 1)$ and $f(n_k + 2) = f((n_k - 2) + 2^2) = f((n_k + 1) + 1)$ divides 5 and 3, so $n_{k+1} - n_k = 2$. By induction the sequence alternates between 1 and 2. In **Case 1.** we already saw that the function equal to 2 on even integers and 1 on odd ones is a solution. We claim that it is not possible to have the opposite parity for all sufficiently large n. Indeed, (3) is contradicted by taking $m = n + 1$ sufficiently large.

Hence, we can assume that $n_k - n_{k-1} \geqslant 3$ for all k and aim for a contradiction. Let

$$m_k = \min\{m \in \mathbb{N} \cup \{\infty\} | m > n_k, f(m) < m - n_k + 1\}$$

and note that $f(m) < m - n_k + 1$ for all $m \geqslant m_k$ and $f(m) = m - n_k + 1$ for all $n_k \leqslant m < m_k$ by (3). Further set $b_k = m_k - n_k \geqslant 3$ (since $n_{k+1} - n_k \geqslant 3$). We also assume that f is not the identity function, so m_1 is finite.

We next prove by induction that $b_{k+1} < b_k$ (and n_{k+1} and m_{k+1} are finite). Assume that this is true for all $k < k_0$. By Bertrand's postulate[1] there exists a prime $b_{k_0} < p < 2b_{k_0}$, where $b_{k_0} = m_{k_0} - n_{k_0}$. But by (3) and the definitions of m_{k_0} and b_{k_0} we have

$$f(p + n_{k_0} - 1) \mid f(m_{k_0} - 1) + f(p - b_{k_0}) = b_{k_0} + (p - b_{k_0}) = p,$$

since $p - b_{k_0} < b_{k_0} \leqslant b_1 < m_1$. Since $f(m) < m - n_{k_0} + 1$ for $m \geqslant m_{k_0}$, we have $f(p + n_{k_0} - 1) = 1$, so $n_{k_0+1} \leqslant p$. Assume for a contradiction that $m_{k_0+1} > n_{k_0+1} + b_{k_0}$.

[1]Other facts on the distribution of primes can be used, but we focus on this one, since it is among the most widely-known.

Fix some $m \leqslant b_{k_0}$ and let $n_{k_0} \leqslant \alpha < m_{k_0}$ and $\beta \leqslant b_1$ be such that $n_k + m = \alpha + \beta^2$. This is indeed possible, since $b_{k_0} + n_{k_0+1} - m_{b_{k_0}} \leqslant 2b_{k_0}$, so that $2\left\lceil\sqrt{b_{k_0} + n_{k_0+1} - m_{b_{k_0}}}\right\rceil \leqslant b_{k_0}$. Hence,

$$m + 1 = f(n_{k_0+1} + m) | f(\alpha) + f(\beta)^2 = n_{k_0+1} - n_{k_0} + m + 1,$$

so $n_{k_0+1} - n_{k_0} < 2b_{k_0}$ is divisible by all integers smaller than b_{k_0}. This is not possible for $b_{k_0} \geqslant 3$ (e.g. using Bertrand's postulate). Hence, b_k form a decreasing sequence which contradicts $b_k \geqslant 3$. Thus, there are no other solutions.

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