# Concours Général de Mathétmatiques "Minko Balkanski"

## SOLUTIONS

3 June 2018

**Solution 1.**

$\boxed{(x, y) \in \{(0,0), (8,6), (-8,-6)\}.}$ The equation can be rewritten as

$$x(x^2 - 37) = y^3 .$$

Note that $x$ and $x^2 - 37$ are either coprime so both cubes or only share the common divisor 37. In the latter case in fact $37^2 \mid x$, so we set $x = 37^2 u$, $y = 37v$ in order to obtain $u(37^3u^2 - 1) = v^3$. Since the two factors are coprime, they need to be both cubes. Then, as $37^3u^2$ and $37^3u^2 - 1$ are both cubes, they are necessarily equal to 0 and $\pm 1$, which leads to $u = 0$ and so $x = y = 0$.

In the former case $x$ and $x^2 - 37$ are both cubes, so $x^2 - 37$ and $x^2$ also are. This is only possible for $x = \pm 8$, so $y = \pm 6$ respectively.

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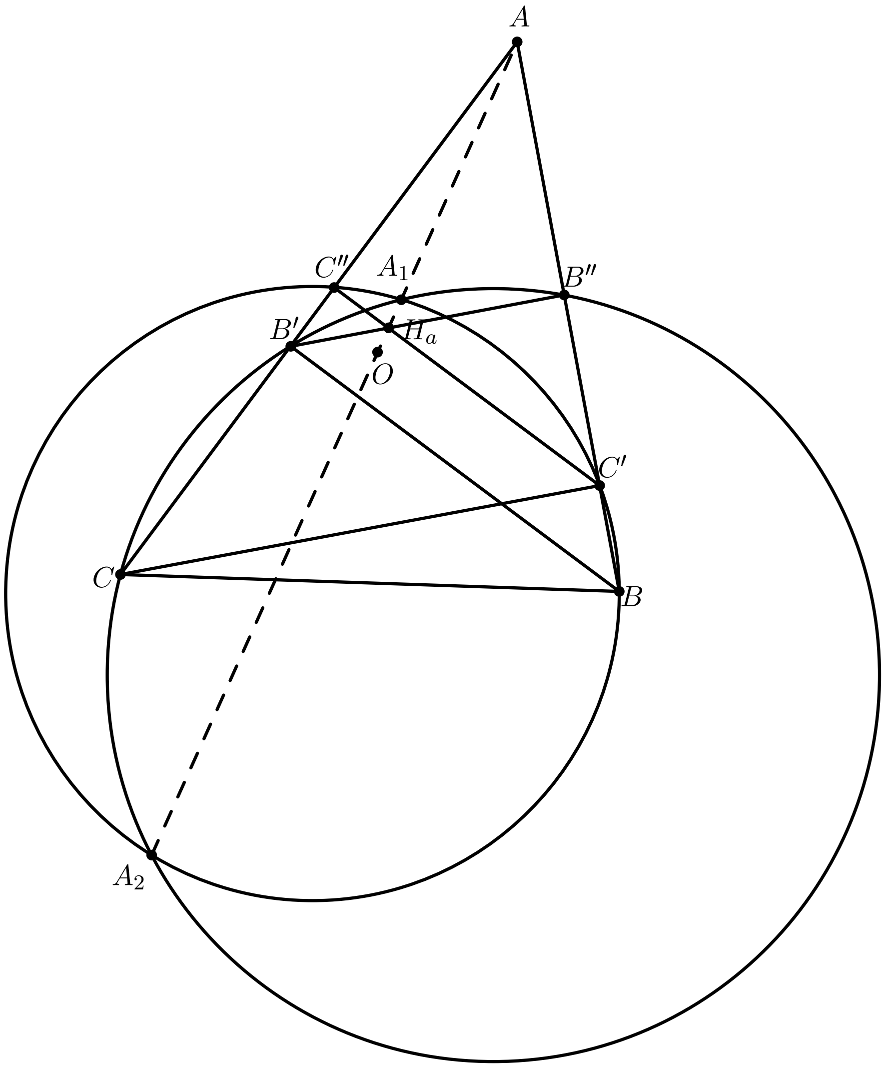
**Solution 2.**

By the radical axes theorem for the circles, circumscribed around $\triangle BC'C''$, $\triangle CB'B''$ and $BCB'C'$ (which is inscribed, because $\angle BB'C = \angle BC'C = 90°$), we obtain that $A_1A_2$ passes through $BC' \cap CB' = A$. Moreover, the orthocentre $H_a$ of triangle $B'C'A$ has equal powers with respect to the circles, circumscribed around $\triangle BC'C''$ and $\triangle CB'B''$, because $B'C''B''C'$ is inscribed, so $H_aB'.H_aB'' = H_aC'.H_aC''$. It follows that $H_a$ is a second point on the radical axis of the two circles, circumscribed around $\triangle BC'C''$ and $\triangle CB'B''$, so this axis is exactly the line $AH_a$. An easy angle chasing allows to verify that $AH_a$ passes through the centre of the circumscribed circle of $\triangle ABC$ and by an analogous reasoning for the vertices $B$ and $C$ we obtain that this is the common point of $A_1A_2$, $B_1B_2$ and $C_1C_2$.

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**Solution 3.**

$\boxed{2^n - 1.}$ Finiteness will follow from our bound, but it can also be proved directly with much less effort (by induction, then assuming the contrary and considering a long interval between two tokens). Note first that the dynamics is reversible, so one can go back to the starting point from every configuration reached by reverting each step. Let $l(n)$ be the largest integer (possibly infinite) on which a token can be placed.

We first show $l(n) \leqslant 2^n - 1$ by induction. The base $n = 1$ is trivial. Let $X \subset \mathbb{N}\backslash\{0\}$ be a configuration reachable with $n$ tokens (we assume without loss of generality that they are all placed on the table). Denote $X = \{x_1^0, \ldots, x_n^0\}$ with $x_1^0 < \ldots < x_n^0$, $X_0 = X \cup \{0\}$ and set $x_0^0 = 0$. By reversibility there exists $i$ such that $x_{i+1}^0 - x_i^0 = 1$ (otherwise there is no possible move). Then $X_1 = X \backslash \{x_{i+1}^0\}$ is also reachable. We denote $X_1 = \{0, x_1^1, \ldots, x_{n-1}^1\}$ still ordered and extend this notation to the $X_k$ to come. More generally, if there are $k$ tokens in Laure's pocket and the other $n-k$ have been placed on $X_k \backslash \{0\}$, by reversibility it is possible to remove some token from the board first – the one on $x_{i+1}^k$, say. But then it is possible to reach $x_{i+1}^k$ without moving any of the tokens on the table using the $k$ tokens from the pocket. By definition this is possible only if $x_{i+1}^k - x_i^k \leqslant l(k) + 1$. We then set $X_{k+1} = X_k \backslash \{x_{i+1}^k\}$. We continue in the same spirit up to $X_n = \{0\}$.

Now notice that

$$\sum_j \left(x_{j+1}^k - x_j^k\right) \leqslant \sum_j \left(x_{j+1}^{k+1} - x_j^{k+1}\right) + l(k) + 1\,,$$

so

$$\max X = x_n = \sum (x_{j+1}^0 - x_j^0) \leqslant \sum (l(k) + 1) = \sum 2^k = 2^n - 1\,.$$

Hence, $l(n) \leqslant 2^n - 1$. It is not hard to follow the equalities in the proof above to see that the case of equality is indeed reached (only) for $X = \{2^{n-1}, 2^{n-1} + 2^{n-2}, \ldots, 2^n - 1\}$ and the strategy above read backwards gives a (recursive) way to reach $X$. More precisely, Laure uses $n-1$ tokens to place the $n$-th on $2^{n-1}$ and removes them afterwards as they were placed. She then uses $n-2$ tokens starting at $2^{n-1}$ to put the last one on $2^{n-1} + 2^{n-2}$, she removes the $n-2$ and so on. Thus, $l(n) = 2^n - 1$.

**Remark.** Let us metion that the problem first appeared in Chung-Diaconis-Graham, 2001.

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**Solution 4.**

For two closed intervals $I$ and $J$ of $[0, 1]$ we write $I \longrightarrow J$ if $f(I) \supset J$ and say that $f(I)$ covers $J$. We know that $f$ acts on the set $\{x_3, f(x_3), f(f(x_3))\}$ as a cycle of length 3. Up to replacing $x_3$ by $f(x_3)$ or $f(f(x_3))$ we may assume that either $x_3 < f(x_3) < f(f(x_3))$ or $x_3 > f(x_3) > f(f(x_3))$. We shall assume that the former chain of inequalities holds and this is without loss of generality, as we may replace $f$ by $\tilde{f}$ defined by $\tilde{f}(x) = 1 - f(1-x)$ (one still has that $1 - x_3$ is of exact period 3). Under this assumption, setting $I_1 := [x_3, f(x_3)]$ and $I_2 := [f(x_3), f(f(x_3))]$, one gets $I_1 \longrightarrow I_2$ and $I_2 \longrightarrow I_1 \cup I_2$. We shall need the following Lemma.

**Lemma.** *Let $I$ and $J$ be two closed intervals such that $I \longrightarrow J$. Then there exists a closed interval $I' \subseteq I$ such that $f(I') = J$.*

*Proof.* Let $J = [a, b]$. Let $x_0, y_0 \in I$ be such that $f(x_0) = a$ and $f(y_0) = b$. For definiteness we assume that $x_0 < y_0$, the other case being treated identically. Set $x = \max\{x' \in I,\ y_0 \geqslant x' \geqslant x_0,\ f(x') = a\}$ and $y = \min\{y' \in I,\ y_0 \geqslant y' \geqslant x,\ f(y') = y\}$, which exist by continuity. We claim that $I' = [x, y]$ does satisfy $f(I') = J$. Indeed, $f(I') \supset J$ by the intermediate value theorem. Moreover, if there exists $y > x' > x$ with $f(x') < a$, then by the intermediate value theorem there exists $y_0 > y > x'' > x'$ such that $f(x'') = a$, which contradicts the definition of $x$. One proves similarly that it is not possible to have $f(y') > b$ for any $x < y' < y$, which completes the proof of the Lemma. $\square$

We will now use this Lemma to prove conclude the solution. Recall that $I_1 \longrightarrow I_2$, $I_2 \longrightarrow I_2$ and $I_2 \longrightarrow I_1$. By the intermediate values theorem we know that $f(x) - x$ must attain the value 0, so there exists a fixed point of $f$. Fix $n \geqslant 2$. We have $I_1 \longrightarrow I_2 \longrightarrow I_2 \longrightarrow \ldots \longrightarrow I_2 \longrightarrow I_1$, where we have $n$ coverings. Thus, by simple induction using the Lemma, we see that there exists a sequence $(J_k)_{0 \leqslant k \leqslant n}$ of closed intervals such that $J_0 = I_1$, $J_n \subseteq I_1$, $f(J_n) = J_{n-1}$ and for all $k < n$ we have $J_k \subseteq I_2$ and $f(J_k) = J_{k-1}$. This implies that $f^{\circ n}(J_n) = J_0 = I_1 \supseteq J_n$, so $f^{\circ n}$ has a fixed point in the interval $J_n$ (we already observed that a continuous function from a closed interval to itself has a fixed point). Moreover, $\forall k \in \{1, 2, \ldots, n-1\}, f^{\circ k}(J_n) \subseteq I_2$, which is disjoint with $I_1$, so the fixed point of $f^{\circ n}$ is a periodic point of exact period $n$ for $f$, which concludes the proof.

**Remark.** The result is a particular case of a theorem of Sharkovsky.

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